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R&D, Subsidies and Productivity

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Technion - Israel Institute of Technology

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Abstract

In this paper we formulate a model that extends the traditional theoretical literature on the optimal management of an R&D project by emphasizing the model's empirical and policy implications. We solve the firm's dynamic problem and use the resulting optimal R&D policy to assess the role of subsidies aimed at stimulating R&D activities. The theoretical framework illustrates that the sign and size of the "additionality effect"—the change in the amount of company-financed R&D caused by the subsidy —depends on the model's parameters and distributional assumptions. It is, therefore, an empirical matter. The model also provides a framework for interpreting anew the empirical association between R&D expenditures and productivity growth found in many micro-level productivity regressions.

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1   Introduction

Economic growth driven by investments in research and development (R&D) is one of the main building blocks of the new models of growth (Aghion and Howitt 1998, Helpman and Grossman 1991). At the same time, the empirical literature reaches the robust conclusion that R&D is associated with productivity gains at the firm and industry level (Griliches, 1998). Yet, "doing R&D"–the process by which investments in research are transformed into new ideas–is a poorly understood concept, especially by economists. This state of affairs is troublesome because policies aimed at promoting R&D activities are consequently based on a weak grasp of the process generating successful R&D outcomes.[1]

In this paper we give a precise meaning to the term "doing R&D" and thus contribute to the micro-foundations of the R&D-driven growth models and empirical work mentioned above. We model the ongoing decisions of a firm manager to engage in *research* and to *develop* the ideas generated by its research. We solve the firm's dynamic problem and use the resulting optimal R&D policy to assess the role of subsidies aimed at stimulating R&D. The model is also used to interpret anew the empirical association between R&D expenditures and productivity growth.

We start from the premise that there is a well defined objective–a new product or a new process–that is known to the firm's manager. The technological process leading to the final outcome can be broken down into stages–these stages being technical problems that need to be successfully addressed in order to make progress towards the final objective. At each stage the firm is involved in "parallel research" (Nelson, 1961), whereby the firm sets up n research teams who are required to come up with alternative solutions to the technical problem associated with the particular stage of the R&D program. After selecting the most profitable among the n alternative solutions, the firm's manager further decides on whether it is worth it to develop such a solution, or to postpone its development until next period in the hope that the research teams will come up with better ideas.

If the proposed solution is implemented and succeeds, the R&D program moves to the next stage. The firm derives returns from the accumulated number of successes. If the solution fails, the firm can try again next period. There is a project reservation cost

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below which the firm will continue with the project and above which the firm will stop its development for the period. Thus, if $n > 0$, the firm never stops doing research but there are periods of more intensive activity when the firm is engaged in the development and implementation of the ideas generated by its research. Otherwise, when no development is undertaken, the R&D staff is just involved in finding better ideas.

The model is rich enough to allow for dynamics, different types of uncertainty and, particularly, to distinguish between research and development activities. Yet it is simple enough to permit an explicit solution for the optimal investment in R&D and for the resulting firm value. The model generates a number of testable predictions and some of these are addressed in the paper.

Under certain conditions, firms facing better markets, or higher probability of success, will have larger research departments and will also develop their ideas more often. Consequently, they will make faster progress in their R&D program. The same is true for firms with a lower cost of setting up research teams. These firms will be more technologically advanced and, even controlling for the technological level, these firms will be more valuable. The model, however, does not imply an unambiguous effect of these parameters on development expenditures. Consequently, the effect of the model's parameters on total R&D expenditures–likely to be the only observed R&D measurement–is also ambiguous, and depends on the parameter configuration and distributional assumptions.

This paper can be seen as continuing the tradition of the old theoretical literature on the optimal management of an R&D project (Kamien and Schwartz, 1971; Lucas, 1971; Grossman and Shapiro, 1981) with special emphasis on the empirical implications of the model. This analysis requires a very detailed description of what constitutes an R&D program: for example, its payoffs, the manner in which "progress" is achieved and the sources of uncertainty. As in the aforementioned papers, we characterize the optimal path of R&D expenditures and derive comparative statics results.

The paper is organized as follows: Section 2 describes the structure of the R&D program, solves the dynamic model explicitly and analyzes its properties. Section 3 introduces R&D subsidies into the model and compares different subsidy policies. The relationship between TFP growth and R&D implied by the model and its implications for the interpretation of empirical estimates of the relationship is taken up in Section 4. Conclusions close the paper.

2   A Model of Research and Development

2.1   The Structure of an R&D Program

Consider a firm that, throughout its lifetime, is involved in the research and development of products or processes within a specific technological field. For example, the firm may be devoted to the research and development of products in the area of digital imaging. This firms' R&D program is implemented by completing a series of *stages.* These stages are technical problems the firm needs to solve in order to advance in its R&D endeavor. As in Dutta (1997), these stages have a natural ordering. That is, each stage has to be completed in a specified order which is known to the firm's manager.

At each stage, the firm is involved in two activities. The first activity is *research*, which consists of R&D teams proposing ideas or solutions to the technical problem faced at each stage. The second activity is *development*, which consists of developing or implementing the "best" proposed solution. Only one idea, if at all, is developed. Throughout the paper, the term R&D *project* comprises both activities–research and development–during a particular stage, while the term R&D *program* refers to the whole sequence of R&D stages or projects.

At the start of each stage, the firm sets up *n identical* research teams that work *independently.* Each research team proposes one solution to the technical problem that needs solving. This "parallel" research approach is one way to cope with the technological uncertainties in R&D, and appears to be quite common in practice.[2] We assume that there are potentially many unknown ways of solving the problem associated with a particular R&D stage. These potential solutions differ in their cost x. The research outcome–the R&D solution–is a random variable because the particular solution that arises from the problem-solving activities of a research team is not known beforehand. In this framework, the outcome of the research phase is a draw from a stationary distribution of costs, denoted by $G(x)$.[3]

There is no guarantee that a solution will work, i.e., the proposed idea may fail to

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[2]A parallel-path strategy was utilized in the famous atomic bomb project, and is quite standard in agricultural and medical research. See Nelson (1961) for other examples and for a formalization of the notion of parallel research. See also Scherer (1984), chapter 6.

[3]In Evenson and Kislev (1976), the solution of the R&D problem also requires a draw from a known distribution of potential solutions. However, the draws are interpreted as different potential technologies rather than different costs of development.

solve the R&D problem and the project will not be completed.[4] In this event, the firm can try again next period. Completing an R&D project means that the firm *successfully* implemented a solution to the problem associated with that particular stage. We denote the probability of success in developing a solution by π and we let this probability increase with the cost of the project. Indeed, we assume, that $0 \leq \pi(x) \leq 1$ is and increasing and concave function of x,[5]

$$\pi(0) = 0, \pi'(x) \geq 0, \pi''(x) \leq 0$$

This means that more expensive solutions are "better" in the sense that they are more likely to succeed. Note that this formulation of the success probability includes the case of a constant π.

The per-period cost of each research team is b.[6] b is thus the cost of sampling from the $G(x)$ distribution. The empirical counterpart to nb could be the wages and equipment of the research staff.

After observing the n solutions, the firm selects the most profitable one, denoted by x^*. [7] x^* reflects the extra costs in materials and equipment associated with implementing the best idea proposed by the R&D staff. The most profitable solution is chosen by trading off increases in the success probability against increases in cost. Higher development expenditures x increase the probability of success but if the cost of development is deemed too high, the firm may decide to postpone development until a better idea arrives. That is, if the research teams were unlucky and drew very unprofitable ideas, the firm may decide not to incur any development expenses at all. If, however, the firm decides to develop the idea, it embarks on the development phase.

Let ω be the state variable of the firm representing the technological progress of the R&D program. If the firm does not implement the proposed solution x^*, it will remain in its current state with probability one. If the proposed solution is implemented and

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[4]In the sample of R&D projects analyzed by Mansfield et al. (1971), the average probability of technical success varied between 52 and 68 percent.

[5]Because the success probability is bounded by 1, concavity is likely to set in. For simplicity, we assume concavity of $\pi(x)$ throughout the support of π.

[6]Linear costs are used for expositional convenience and because it corresponds to our numerical examples (see below). The analysis can be trivially extended to the case of increasing, convex, costs of research.

[7]If all sampled solutions differ only in their cost, i.e.,if $\pi(x)$ is flat, the most profitable solution would also be the cheapest one in the sample.

succeeds, the firm advances one research stage. Thus, given today's technological state ω, next period's state ω' will either remain at ω or increase to $\omega + 1$. If the firm decides to develop, the firm's state variable evolves according to

$$\omega' = \begin{cases} \omega \text{ with probability } 1 - \pi(x^*) \\ \\ \omega + 1 \text{ with probability } \pi(x^*). \end{cases} \tag{1}$$

Thus, ω represents the number of accumulated successes in the R&D program. As in Ericson and Pakes (1995), the firm is assumed to market and make profits on the product using the current technological state ω of the R&D program. For tractability, it is assumed that current returns are linear

$$\phi(\omega) = \phi\omega \tag{2}$$

In Section 4, $\phi\omega$ is interpreted as the profits from producing output using a Cobb-Douglas production function in conventional inputs and capital-enhancing technology level ω, gross of R&D costs. Prices are assumed to be constant and known to the firm.

Note that neither the probability of successful development $\pi(x)$ nor the distribution of (costs of) ideas $G(x)$ depend on the technological state ω. These assumptions simplify the analysis without hindering the main message of the paper. Their drawback is that the resulting dynamics are driven solely by the future returns accruing to accumulated R&D successes.

Note also that more research–a higher n–does not increase the probability of success π, conditional on development being undertaken. However, as will be shown below, a higher n implies that a profitable solution is more likely to be found and, consequently, it raises the probability of development.

The present model differs from traditional models of R&D (e.g., Kamien and Schwartz, 1971; Lucas, 1971; Grossman and Shapiro, 1981) in several dimensions. First, traditional model do not explicitly distinguish between research (n) and development (x^*). Second, in traditional models the magnitude of the R&D outcome–the R&D step–is stochastic, whereas in the present model the R&D step is known but its cost is uncertain: the firm knows exactly where it is heading and what needs to be done at each R&D stage and only needs to find a way of doing it in the most profitable way. Third, in traditional models more R&D effort always leads to more outcomes or to a higher probability of success, whereas in the present model development expenditures may be so high that development is postponed. Fourth, the model developed in this paper does not impose a

terminal stage at which point a discounted infinite stream of returns is collected. Lastly, the present model introduces two types of uncertainty: a technical-related uncertainty in the form of π and a cost-related uncertainty in the form of $G(x)$.

2.2 The Firm's Dynamic Problem

The firm's manager chooses the optimal number of research teams n and decides on whether or not to pursue development in order to maximize the expected discounted stream of per-period profits.

In each period, and in any state ω, the firm chooses the number of research teams n, takes n draws from $G(x)$, and collects $\phi\omega - nb$. After selecting x^*–the most profitable idea among the n proposals[8]–the firm decides whether to develop the solution embodied in the realized x^* or to suspend the development of the R&D project. If the project is developed, the firm pays x^* and moves to $\omega + 1$ with probability $\pi(x^*)$ or stays at ω with probability $1 - \pi(x^*)$. If no development is undertaken, the firm remains at ω and waits until next period in the hope of getting a better x^*.

Let $V(\omega)$ be the maximal value of the expected discounted stream of per-period profits when the firm is in technological state ω. Let $V_1(\omega, n, x)$ be the maximal value of the R&D program at state ω when the firm samples n solutions and the proposed solution costs x, but the firm decides *not* to develop the R&D project. And let $V_2(\omega, n, x)$ be the maximal value associated with the decision to develop (continue) the R&D project. $V_1(\omega, n, x)$ and $V_2(\omega, n, x)$ are defined by

$$V_1(\omega, n, x) = V_1(\omega, n) = \phi\omega - nb + \beta V(\omega)$$

$$(3)$$

$$V_2(\omega, n, x) = \phi\omega - nb - x + \beta\pi(x)V(\omega + 1) + \beta(1 - \pi(x))V(\omega)$$
$$= V_1(\omega, n) + \beta\pi(x)\left[V(\omega + 1) - V(\omega)\right] - x$$

where $0 \leq \beta < 1$ is the discount factor.[9]

Expression (3) says that when the firm does *not* develop the R&D project, and ω remains unchanged in the next period, the firm's value equals the current net returns

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[8]This is analogous to Stigler's (1961) model of searching for the lowest price from a known distribution of prices.

[9]We assume there are no liquidity constraints: capital markets allow the firm to borrow as needed at the discount rate β.

$\phi\omega - nb$ plus the maximal value of the expected discounted stream of per-period profits when the firm is in technological state ω. If the firm pursues the project, a cost x is incurred and the firm moves to state $\omega + 1$ with probability $\pi(x)$, or remains in state ω with probability $1 - \pi(x)$.

Note that V_2 equals V_1 plus the *continuation value*

$$C(\omega, x) \equiv \beta\pi(x)\left[V(\omega + 1) - V(\omega)\right] - x \tag{4}$$

$C(\omega, x)$ equals the expected discounted stream of additional profits resulting from the development of the ideas at stage ω. Because $C(\omega, x)$ refers to additional benefits for a given cost x, the continuation value does not depend on n, the number of ideas sampled. Research expenditures are sunk.

Given n, the most profitable project among the n proposals $x_1, \ldots, x_n$ is the one giving the highest value $V_2(\omega, n, x)$ or, equivalently, the highest continuation value. Thus, x^* is defined implicitly by

$$
\begin{aligned}
V_2(\omega, n, x^*) &= Max\left\{V_2(\omega, n, x_1), \ldots, V_2(\omega, n, x_n)\right\} \\
&= V_1(\omega, n) + Max\left\{C(\omega, x_1), \ldots, C(\omega, x_n)\right\} \\
&= V_1(\omega, n) + C(\omega, x^*)
\end{aligned}
$$

where $C(\omega, x^*) = Max\left\{C(\omega, x_1), \ldots, C(\omega, x_n)\right\}$.

$V(\omega)$, the maximal value of the R&D program at ω, satisfies the following functional equation

$$V(\omega) = \underset{n}{Max}\, EMax\left\{V_1(\omega, n), V_2(\omega, n, x^*)\right\} \tag{5}$$

The inner maximization refers to the development decision–develop only if its expected value is larger than that of not developing–given x^*, while the outer maximization refers to the choice of n which affects the probability distribution of x^*.

Using the expression for V_1 and V_2 from (3) we can also write

$$
\begin{aligned}
V(\omega) &= \underset{n}{Max}\, \frac{1}{1-\beta}\left[\phi\omega - nb + EMax\left\{0, C(\omega, x^*)\right\}\right] \tag{6} \\
&= \underset{n}{Max}\, \frac{1}{1-\beta}\left[\phi\omega - nb + E\left(C(\omega, x^*)|C(\omega, x^*) \geq 0, n\right)P_n(C(\omega, x^*) > 0)\right]
\end{aligned}
$$

where $P_n(C(\omega, x^*) > 0)$ is the probability that $C(\omega, x^*)$ is positive given a sample of n proposals.

The firm chooses n to maximize the discounted stream of current net returns, $\frac{\phi\omega - nb}{1-\beta}$, plus the expected continuation value, given that it is positive, weighted by the probability that it is positive. As shown in Appendix 1, the probability distribution of $C(\omega, x^*)$ depends on n because it is the continuation value of the most profitable solution in a sample of n solutions.

The expected value of the maximum between zero and the continuation value, the $EMax$ term in (6), plays an important role in the solution. We will, therefore, denote it by

$$EC(\omega, n) = E\left(C(\omega, x^*)|C(\omega, x^*) \geq 0, n\right) P_n(C(\omega, x^*) > 0) \qquad (7)$$

2.2.1 Optimal Development

Note from (5) that, given n, the firm chooses to develop the solution x^* if $V_2(\omega, n, x^*) > V_1(\omega, n)$, i.e., if the continuation value is positive, and chooses to wait until next period if $V_2(\omega, n, x^*) < V_1(\omega, n)$, i.e., if the continuation value is negative.

We can show the development decision graphically by examining the continuation function (4). For given ω, the function $C(\omega, x)$ starts at $C(\omega, 0) = 0$ and, provided

$$C'(\omega, 0) = \beta \left[V(\omega + 1) - V(\omega)\right] \pi'(0) - 1 > 0 \qquad (8)$$

$C(\omega, x)$ increases until it attains a maximal value $\widehat{C} = C(\omega, \widehat{x}(\omega))$ at $\widehat{x}(\omega)$.[10] Thereafter, $C(\omega, x)$ declines towards the negative values (see Figure 1).[11] We will assume that (8) holds at every ω to ensure that the development option is not ruled out a-priori.

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[10]x^* is the value of x_i in the sample of size n that has the highest continuation value, while $\widehat{x}(\omega)$ is the maximizer of the continuation function. Thus $C(\omega, x^*) \leq C(\omega, \widehat{x}(\omega))$ and equality is attained when one of the sample realizations equals $\widehat{x}(\omega)$.

[11]The range of the continuation function is $(-\infty, \widehat{C})$.
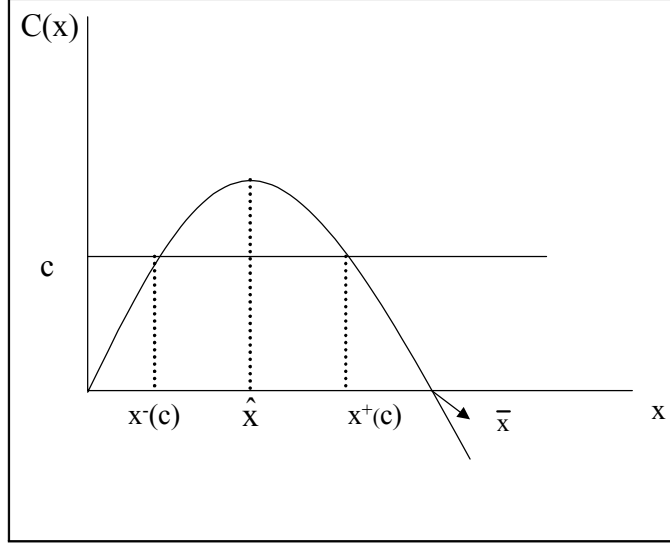
Figure 1: Continuation Value $C(\omega, x)$

It follows that there exists a positive value of x at which the continuation value is zero. This *reservation cost,* denoted by $\overline{x}(\omega)$, is defined by

$$0 = \beta\pi(\overline{x}(\omega))\left[V(\omega + 1) - V(\omega)\right] - \overline{x}(\omega)$$

Clearly, $C(\omega, x)$ is positive for $x < \overline{x}(\omega)$ and negative for $x > \overline{x}(\omega)$. The firm therefore follows a reservation cost policy:

$$\text{develop the project if } x^* < \overline{x}(\omega) \Longleftrightarrow C(\omega, x^*) > 0$$
$$\text{suspend the project if } x^* > \overline{x}(\omega) \Longleftrightarrow C(\omega, x^*) < 0$$

When x^* equals $\overline{x}(\omega)$ the firm is indifferent between postponing and developing the project: the continuation value is zero.

2.2.2 Optimal Research

The solution to the maximization problem in (6) requires the conditional expectation of $C^* \equiv C(\omega, x^*)$. In Appendix 1, we show that its distribution function is

$$F_{\omega,n}(c) = \left[1 + G(x^-(c)) - G(x^+(c))\right]^n \tag{9}$$

where $x^-(c)$ and $x^+(c)$ are, respectively, the minimal and maximal roots of the equation $C(\omega, x) = c$, and $x^-(c) \leq \widehat{x}(\omega) \leq x^+(c)$ (see Figure 1).

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Note that $F_{\omega,n}(c)$ decreases with n; the larger the size of the research department, the higher the probability of coming up with more profitable solutions and, thus, the higher the expected continuation value. Note also that $F_{\omega,n}(c)$ depends on ω, because $x^-(c)$ and $x^+(c)$ depend on $V(\omega+1) - V(\omega)$.

The probability of a positive continuation value–the probability of development–is $1 - F_{\omega,n}(0)$. Because

$$1 - F_{\omega,n}(0) = 1 - \left[1 + G(x^-(0)) - G(x^+(0))\right]^n \tag{10}$$
$$= 1 - [1 + G(0) - G(\overline{x}(\omega))]^n = 1 - [1 - G(\overline{x}(\omega))]^n$$

the probability of development equals the probability that the lowest x_i is less than or equal to the reservation cost $\overline{x}(\omega)$. This should indeed be the case because the two events are equivalent.

The optimal number of research teams is found by equating the marginal benefit (MB) of n to its marginal cost, b. MB_n is the difference in the expected continuation value when the firm moves from $n-1$ to n research teams

$$MB_n = EC(\omega, n) - EC(\omega, n-1)$$

$$= \frac{1}{1-\beta}\left(\int_0^{\widehat{C}} c\, dF_{\omega,n}(c) - \int_0^{\widehat{C}} c\, dF_{\omega,n-1}(c)\right) \tag{11}$$

$$= \frac{1}{1-\beta}\left(\int_0^{\widehat{C}} F_{\omega,n-1}(c)\left(G(x^+(c)) - G(x^-(c))\right) dc\right)$$

using integration by parts and $F_{\omega,n}(\widehat{C}) = 1$.

From the last row in (11) we have that $MB_n > 0$, that MB_n is a decreasing function of n, and that $\lim_{n\to\infty} MB_n = 0$ (see Figure 2). MB_n also depends on ω but this is left implicit in the notation.
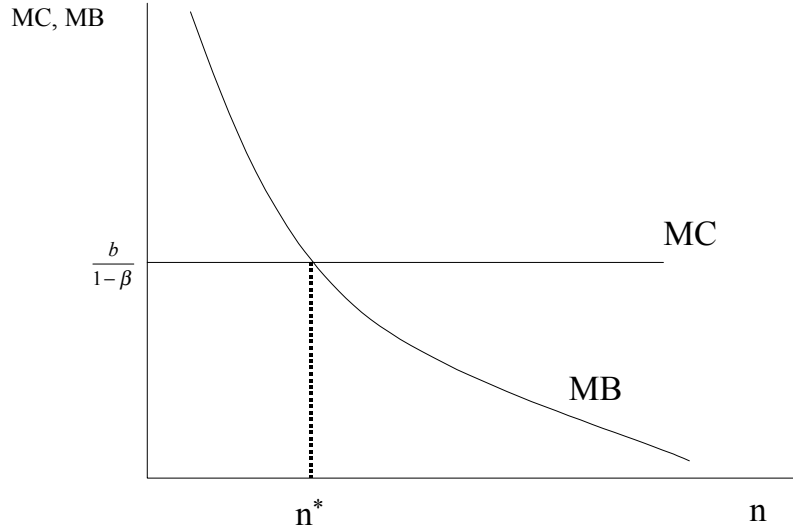
Figure 2: Optimal n

The (unique) optimal n is an integer satisfying.

$$MB_n \geq \frac{b}{1 - \beta} > MB_{n+1} \tag{12}$$

This condition says that the marginal benefits from setting up the nth research team should be larger than its cost, but the marginal benefits of the $n^{th} + 1$ team should be below its setup cost. Because MB_n depends on $F_{\omega,n}(c)$ which depends on $V(\omega + 1) - V(\omega)$, the optimal number of research teams may vary with the state ω.

Clearly, as long as $b > 0$, it is not optimal to let n increase without bounds. Thus, the optimal n is finite but can be zero. To ensure a positive n, the marginal benefit of the first draw must be larger than its marginal cost

$$MB_1 = \frac{1}{1 - \beta} \int_0^{\widehat{C}} \left(G(x^+(c)) - G(x^-(c)) \right) dc \geq b \tag{13}$$

Requirement (13) says that, given a distribution of development costs G, the fixed costs of research b should not be "too high". Loosely speaking, the MB curve should intercept the vertical axis at a higher value than b.[12]

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[12]Because the expected continuation value at $n = 0$ is zero, we also have

$$MB_1 = \frac{1}{1 - \beta} E(C^*|C^* \geq 0, n = 1) G(\overline{x}(\omega))$$

which says that it pays to set up at least one research team when the expected continuation value from a single x draw is larger than its cost.

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An interesting issue that has attracted attention in the empirical literature on R&D is the characterization of firms that choose to engage in R&D (see Bound et al. (1984)). In our model, a firm that engages in R&D is a firm that has a positive n. From (13) we see that firms with lower $b's$ are more likely to find R&D a rewarding activity and will, therefore, have a positive n. Differences in b across firms may arise from differential access to financial capital. The model therefore implies that firms facing lower marginal costs of obtaining funds are more likely to engage in R&D. Indeed, Bond, Harhoff and Van Reenen (1999) find that, for British firms, financial constraints affect the decision to engage in R&D rather than the level of R&D spending by participants.

In order to trace the effect of changes in the model parameters (π, ϕ, and b) on the optimal number of research teams or to trace the evolution of R&D expenditures and the technological state ω over time, it is necessary to know some features of the value function. The following proposition is proved in Appendix 2.

Proposition 1 *The value function is linear in ω. It is given by*

$$V(\omega) = \frac{1}{(1-\beta)} \left(\phi\omega - n^*b + \int_0^{\frac{\beta\phi}{1-\beta}\pi(\widehat{x})-\widehat{x}} c \left[1 + G(x^-(c)) - G(x^+(c)) \right]^{n^*} dc \right) \quad (14)$$

where n^* satisfies (12) and $\widehat{x}$ satisfies $\phi\beta\pi'(\widehat{x}) = 1 - \beta$.

The reservation cost project is $\overline{x} = \frac{\beta\pi(\overline{x})\phi}{(1-\beta)}$, while the continuation value is $C(x_i) = \frac{\beta\pi(x_i)\phi}{(1-\beta)} - x_i$.

The key assumptions allowing us to derive an analytical solution for $V(\omega)$ are the linearity of the current return function $\phi(\omega)$ in (2), and the non-dependence of π on ω.[13] These assumptions ensure that the value function is linear in ω, and this implies that neither the continuation function nor the distribution function of $C(x^*)$ depend on ω. Because the choice of n is determined by the expected continuation value, *optimal n does not depend on ω either.*

As mentioned above, there is no guarantee that a strictly positive n exists: under some parameter configurations, it may be more profitable to *never* engage in R&D. However, if at some point it is profitable to do research, i.e., $n > 0$, then the firm will

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[13]Up to this point the model could have been trivially generalized to allow for concave returns $\phi(\omega)$, increasing-in-ω success probability $\pi(\omega, x)$, and increasing, convex, research costs $b(n)$. The continuation value would simply be $\beta\pi(\omega, x) [V(\omega + 1) - V(\omega)] - x$.

always be engaged in research. Furthermore, the firm will be doing the same amount of research in every state ω. Nevertheless, because of the randomness in x^*, the firm will not always develop its ideas, although the probability of development is the same in every state ω. Thus, R&D expenditures consist of a constant expenditure on research, nb, and a random expenditure on development. This implies a persistence in R&D expenditures over time which is consistent with the empirical findings (see Lach and Schankerman,1989; Mulkay et al., 2000).

An intuitive understanding of the value function follows from the analysis of the value of not developing a solution at state ω. From (3), (10) and Proposition 1, we have

$$V_1(\omega) = \frac{1}{(1-\beta)} \left(\phi\omega - bn^* + \beta E \left(\frac{\beta\pi(x^*)\phi}{(1-\beta)} - x^* | x^* \leq \overline{x} \right) \left(1 - [1 - G(\overline{x})]^{n^*}\right) \right)$$

using the equality $\int_0^{\widehat{C}} c \, dF_n(c) = E\left(C(x^*)|C(x^*) \geq 0, n\right)(1 - F_n(0))$.

The first term in $V_1(\omega)$ is the discounted present value of the number of accumulated successes net of research costs, $\frac{\phi\omega - bn^*}{1-\beta}$. The expected continuation value takes account of the expected revenues and development costs from period $t+1$ onwards. A success occurs with probability $\pi(x^*)\left(1 - [1 - G(\overline{x})]^{n^*}\right)$ and yields returns ϕ forever. Thus, $\beta^2 \frac{\left(1 - [1 - G(\overline{x})]^{n^*}\right) E\pi(x^*)\phi}{(1-\beta)}$ is the expected discounted present value of a single expected success in period $t+2$ (note that the expectation operator uses the distribution function of x^*). The expected discounted development cost in period $t+1$ is $\beta E\left[x^*|x^* \leq \overline{x}\right]$ with probability $1 - [1 - G(\overline{x})]^{n^*}$ and zero with probability $[1 - G(\overline{x})]^{n^*}$. Adding up the stream of revenues and costs from period $t+1$ onwards and discounting yields the second term in $V_1(\omega)$. The value of developing the project in period t, $V_2(\omega, x^*)$, equals $V_1(\omega)$ plus the continuation value associated with x^*.

In sum, the firm pursues an R&D program whose state of progress is given by the number of accumulated successes ω. In each period, the firm has to solve a technical problem associated with the current project. The proposed solution to the technical problem costs x^* which is the most profitable solution among n proposals drawn from some distribution G. If the proposed solution is implemented and works, the R&D program moves to the next stage. If the solution fails, the firm can try again next period. The firm chooses the amount of research–the number of draws on $x-$and based on the realized value of the most profitable proposal, it decides whether or not to develop the proposed solution. It will opt not to develop the proposed solution when the solution is

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too expensive. There is a project reservation cost below which the firm will continue with the project and above which the firm will stop its development for the period. Thus, if $n > 0$, the firm never stops doing research but there are periods of more intensive activity when the firm is engaged in the development and implementation of the ideas generated by its research. Otherwise, when no development is undertaken, the R&D staff is just involved in finding better ideas. In this latter case, R&D expenditures are only nb.

2.3   Comparative Statics

Let us analyze first the effect of the model's parameters on the value function. From (14), and using an envelope theorem it is clear that $V(\omega)$ decreases with b, the marginal cost of a research team. The effect of changes in ϕ on $V(\omega)$ operate through the current returns and through the continuation function C. Changes in π affect only the continuation function and their effect is essentially the same as the effect of changes in ϕ. More successful firms and firms facing better markets (higher ϕ) have higher continuation values at any x (and ω). The following proposition is proved in Appendix 3.

Proposition 2 $V(\omega)$ *decreases with b, and increases with ϕ and π.*

The model parameters also affect the distribution of continuation values and, therefore, the probability of development. Note that

$$\frac{\partial F_n(c)}{\partial \phi} = -nF_{n-1}(c)\frac{\partial H(c)}{\partial \phi}$$

where $0 \leq H(c) = G(x^+(c)) - G(x^-(c)) \leq G(\overline{x})$ is a decreasing function of c.

In the proof of Proposition 2, we show that $\frac{\partial H(c)}{\partial \phi} > 0$. Thus, the higher ϕ (or π), the distribution of continuation values gets better in the sense that the probability of obtaining higher $c's$ increases. We have

Corollary 1 $F_n(c)$ *decreases with ϕ (and π) at every c.*

This means that more successful firms, and firms facing better markets, not only have higher continuation values, but also higher probabilities of obtaining them. This implies, of course, that these firms are more likely to engage in development of their research ideas.

Consider now the effect of a change in b, the marginal cost of research, on the number of research teams n. Because n is discrete–there is no meaning to half a draw– small changes in b, or in any of the model's parameters, may not affect the optimal n,

i.e., the inequalities of the first order condition still hold at the same n after a small change in b. This implies that optimal n is a step-function of b. Because the b curve in Figure 2 shifts upward and the MB curve is not affected, the optimal n decreases. Graphing the optimal n against b results in a step function decreasing in jumps of size one.

As mentioned above, an increase in ϕ (or in π) increases the continuation value $C(x)$ at any x but its effect on the marginal benefit MB_n is not immediately clear. The firm will increase n if the *marginal* benefit of a research team MB_n increases–if the MB curve shifts up in Figure 2. But the assumptions so far do not guarantee that this will occur.[14]

Because the marginal benefit is the difference in conditional expected continuation values when n increases by one unit, equation (11), the change in MB_n resulting from an increase in ϕ depends upon its effect on the expected continuation value given n and given $n - 1$ research teams, $EC(n)$ and $EC(n-1)$. If these two conditional values increase by the same amount then MB is not affected by ϕ. If the effect on the expected continuation value is larger the larger is n, then MB_n increases with ϕ prompting an increase in n. In this case, n and ϕ are, loosely speaking, "strategic complements". Changes in ϕ affect positively the marginal benefit of n and, given decreasing marginal benefits, n has to increase for an optimum to be achieved. If n were a continuous variable, n will increase with ϕ if $\frac{\partial^2 EC(n,\phi)}{\partial n \partial \phi} > 0$. If, on the other hand, increases in ϕ increase $EC(n-1)$ by more than they increase $EC(n)$, MB_n will decrease and so will optimal n.

In Appendix 4, we proof the following proposition

Proposition 3 *If* $1 - F_n(c; \phi)$ *exhibits strictly increasing differences in* (n, ϕ) *then* n *increases with* ϕ.

A similar proposition holds for changes in π instead of in ϕ.

Proposition 3 gives a sufficient condition to ensure monotonic comparative statics. The condition is that the probability that $C(x^*)$ is greater than any c exhibits strictly increasing differences in (n, ϕ). Intuitively, if the increase in the probability of getting large $C's$ resulting from an increase in n is larger the larger is ϕ, then it pays to incur

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[14]For example, suppose that after the increase in ϕ the new maximizer of $C(x)$ is found in a segment of the support of x whose probability is very high. Then, at the higher ϕ, it may pay the firm to decrease n and save the cost b.

the extra cost b and increase n when ϕ increases.[15] Thus, for monotonic comparative statics to hold in this model, the distribution function $G(x)$ and the success probability function $\pi(x)$ should be restricted to the set of functions guaranteeing that $1 - F_n(c; \phi)$ exhibits strictly increasing differences in (n, ϕ). That is, without imposing restrictions on these functions, the amount of research may not be unambiguously increasing in the market and technological success parameters.

This condition implies that $EC(n, \phi)$ exhibits strictly increasing differences which means that for $\phi_2 > \phi_1$ (see Appendix 4),

$$
\begin{aligned}
MB_n(\phi_2) & = EC(n, \phi_2) - EC(n - 1, \phi_2) > \\
& \quad EC(n, \phi_1) - EC(n - 1, \phi_1) \\
& = MB_n(\phi_1)
\end{aligned}
$$

Development expenditures, denoted by d, are either zero if $x^* > \overline{x}$ or x^* if $x^* \leq \overline{x}$,

$$
d = \begin{cases} 0 & \text{if } x^* > \overline{x} \\ \\ x^* & \text{if } x^* \leq \overline{x} \end{cases} \tag{15}
$$

In order to characterize expected development expenditures we first need to derive the density function of x^*. We argue as follows. Let $x < \widehat{x}$. The probability that in a particular sample of size n, the most profitable project costs x is equal to $g(x)$ times the probability that the other $n - 1$ draws are less profitable, $[1 - G(x^+(C(x))) + G(x)]^{n-1}$, times n. We proceed analogously for $\widehat{x} \leq x \leq \overline{x}$ and for $x > \overline{x}$. Then,

$$
g_{x^*}(x; n) = \begin{cases} ng(x)\left[1 - G(x^+(C(x))) + G(x)\right]^{n-1} & x < \widehat{x} \\ \\ ng(x)\left[1 - G(x) + G(x^-(C(x)))\right]^{n-1} & \widehat{x} \leq x \leq \overline{x} \\ \\ ng(x)\left[1 - G(x)\right]^{n-1} & \overline{x} < x \end{cases} \tag{16}
$$

Expected development expenditures d are, therefore,

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[15] If n were a continuous variable, this condition is implied by an everywhere positive cross-partial derivative $\frac{\partial(1 - F_n(c; \phi))}{\partial n \partial \phi}$.

$$E(d) = E(x^*|x^* \leq \overline{x})P(x^* \leq \overline{x}) = \int_0^{\overline{x}} x g_{x^*}(x; n) dx$$

(17)

$$E(d|d > 0) = E(x^*|x^* \leq \overline{x}) = \int_0^{\overline{x}} x \frac{g_{x^*}(x; n)}{1 - [1 - G(\overline{x})]^n} dx$$

These complicated expressions are difficult to characterize analytically. We therefore resort to a parametric example.

2.4   A Parametric Example

We illustrate the comparative static results using a piece-wise linear functional form for the success probability,

$$\pi(x) = \begin{cases} a_0 x & 0 \leq x \leq \frac{1}{a_1} \\ \frac{a_0}{a_1} & x \geq \frac{1}{a_1} \end{cases}$$

(18)

for $a_1 \geq a_0 \geq 0$.

The π function increases linearly a rate a_0 until x reaches $\frac{1}{a_1}$. Thereafter, π remains constant at $\frac{a_0}{a_1}$. The continuation value is

$$C(x) = \begin{cases} \frac{\beta\phi}{1-\beta} a_0 x - x & 0 \leq x \leq \frac{1}{a_1} \\ \frac{\beta\phi}{1-\beta} \frac{a_0}{a_1} - x & x \geq \frac{1}{a_1} \end{cases}$$

Figure 3 plots the success probability and continuation value functions.

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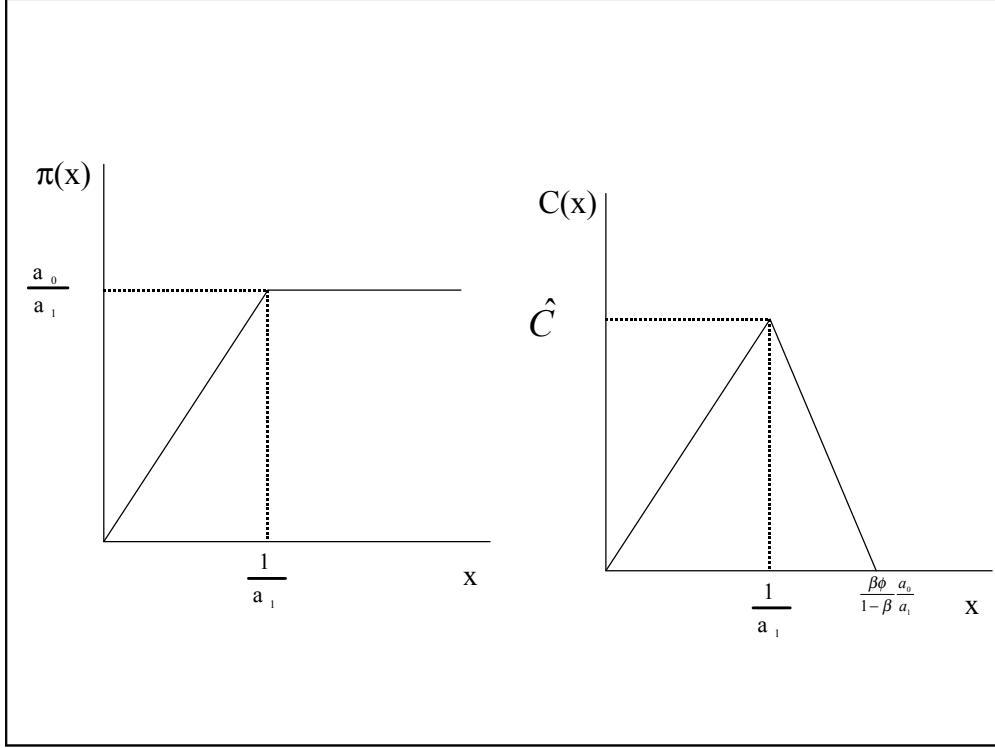
Assumption (8) implies that $\frac{\beta\phi}{1-\beta}a_0 > 1$. The highest continuation value is obtained at $\widehat{x} = \frac{1}{a_1}$, which does not depend on ϕ. The continuation value at $\widehat{x}$ is

$$\widehat{C} = C(\widehat{x}) = \frac{\beta\phi}{1-\beta}\frac{a_0}{a_1} - \frac{1}{a_1} = \frac{\beta\phi a_0 - (1-\beta)}{(1-\beta)\,a_1}$$

The reservation cost is

$$\overline{x} = \frac{\beta\phi}{1-\beta}\frac{a_0}{a_1}$$

while the two roots of $C(z) - c = 0$ are

$$x^-(c) = \frac{(1-\beta)}{\beta\phi a_0 - (1-\beta)}c$$

$$x^+(c) = \frac{\beta\phi}{1-\beta}\frac{a_0}{a_1} - c$$

We computed the value function $V(\omega)$ in Proposition 1 using (18) and three different distributions for x (uniform, exponential and lognormal) maintaining the same mean of x in each distribution. For any given distribution, we first solved the non-linear

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equation (12) for the optimal number of research teams. The integral part of the non-linear equation is then evaluated using a 21-point Gauss-Konrod quadrature rule and the cumulative distribution function of the assumed distribution of x. The integral is evaluated in each iteration of Muller's method for solving non-linear equations. The optimal number of research teams does not vary with ω implying that the non-linear equation needs to be solved only once. The optimal number of research teams is then substituted into $V(\omega)$ and the integral part of the value function is evaluated using the same Gauss-Konrod quadrature rule. Again, the integral part of the value function needs to be solved only once since it does not vary with ω.[16]

The parameters used in all the computations are $\beta = 0.96$, $a_0 = 0.08$ and $a_1 = 0.10$, while we varied the values of b (from 0.10 to 1.10) and ϕ (from 0.8 to 1.8). The most profitable project is constant at $\widehat{x} = \frac{1}{0.10} = 10$. The mean of x was set to 40 implying that the upper limit in the uniform distribution is 80, and that the exponential parameter is $\frac{1}{40}$. For the lognormal distribution, we put $\sigma = 1$ and $\mu = 3.189$. We computed the value function for ω ranging from 1 to 100.

Table 1 presents the effect of changing the key parameters of the model on the optimal number of research teams n. Because changes in π and ϕ have, qualitatively, the same effects we presents comparative static results only with respect to changes in ϕ. As expected, n increases with decreases in b and, at least in these computations, it

---

[16]For some distributions, however, we can solve explicitly for the value function. For example, when $G(x)$ is the *uniform* distribution on $[0, \tau]$ with $\tau > \overline{x}$ we obtain the following expression for the value function (14)

$$V(\omega) = \frac{\phi}{1-\beta}\omega + \frac{1}{1-\beta}\left(-nb + \frac{\overline{x} - \frac{1}{a_1}}{(n+1)}\left[n+1 - \frac{\tau}{\overline{x}}\left\{1 - \left(1 - \frac{\overline{x}}{\tau}\right)^{n+1}\right\}\right]\right)$$

where n is the optimal n satisfying (12).

The distribution function for C^* is

$$F_n(c) = \left(1 - \frac{\beta\phi a_0}{(1-\beta)a_1}\frac{1}{\tau} + \frac{\beta\phi a_0}{\beta\phi a_0 - (1-\beta)}\frac{1}{\tau}c\right)^n = \left(1 - \frac{\overline{x}}{\tau} + \frac{\overline{x}}{\widehat{C}\tau}c\right)^n$$

also increases with ϕ.[17]

Table 1: Comparative Statics on n

		Number of Research Teams (n)		
b	ϕ	Uniform	Lognormal	Exponential
Change in b				
0.1	1.0	18	13	15
0.3	1.0	9	7	8
0.5	1.0	5	5	5
0.8	1.0	2	3	3
1.1	1.0	1	2	2
Change in ϕ				
0.5	0.8	1	3	3
0.5	1.0	5	5	5
0.5	1.2	7	6	6
0.5	1.5	9	6	7
0.5	1.8	10	7	8

Table 2 presents the effect of changes in b and ϕ on expected development expenditures (17). We distinguish between the average of *positive* development expenditures, $E(d|d > 0)$, and the average that includes zero development expenditures as well, i.e., when the project is not developed. In order to compute these averages, we simulated outcomes for 3,000 identical firms over 100 investment periods. In each period, we drew n $x\prime s$ and kept x^*. If $x^* \leq \overline{x}$ we set $d = x^*$; otherwise we put $d = 0$. The entries in Table 2 are averages over the 3,000 firms and over all 100 $\omega\prime s$ (300,000 simulated observations).

---

[17]Under the uniform distribution assumption we obtain,

$$MB_n = \frac{1}{1-\beta}\frac{\left(1 - \frac{1}{\overline{x}a_1}\right)\tau}{n(n+1)}\left(1 - (1 - \frac{\overline{x}}{\tau})^n(1 + \frac{\overline{x}}{\tau}n)\right)$$

Note that MB_n increases with $\overline{x}$, which increases with ϕ.

Table 2: Comparative Statics on $E(d)$

			$E(d\|d > 0)$			$E(d)$			$P(d > 0)$		
			(1)			(2)			(3)		
b	ϕ	$\overline{x}$	U	LN	EXP	U	LN	EXP	U	LN	EXP
Change in b											
0.1	1.0	19.2	9.82	9.90	9.76	9.75	9.89	9.75	0.99	0.99	0.99
0.3	1.0	19.2	9.74	9.98	9.49	8.91	9.73	9.28	0.91	0.97	0.98
0.5	1.0	19.2	9.67	10.11	9.27	7.21	9.37	8.42	0.75	0.93	0.91
0.8	1.0	19.2	9.62	10.32	9.07	4.06	8.18	6.92	0.42	0.79	0.76
1.1	1.0	19.2	9.61	10.47	8.95	2.31	6.79	5.52	0.24	0.65	0.62
Change in ϕ											
0.5	0.8	15.4	7.71	8.95	7.56	1.49	6.19	5.17	0.19	0.69	0.68
0.5	1.0	19.2	9.67	10.11	9.27	7.21	9.37	8.42	0.75	0.93	0.91
0.5	1.2	23.0	11.05	10.79	10.41	10.01	10.58	10.08	0.91	0.98	0.97
0.5	1.5	28.8	12.30	11.63	11.55	12.08	11.56	11.48	0.98	0.99	0.99
0.5	1.8	34.6	13.03	11.94	12.16	12.98	11.93	12.15	0.99	0.99	0.99

A decrease in b affects expected development expenditures d through the increase it induces in n. As n increases, two things happen. First, the firm is more likely to engage in development (see equation (10)) and this increases average development expenditures through the increase in the fraction of projects being developed (see the second and third columns in the upper panel of Table 2). Second, the firm is more likely to draw an x^* closer to $\hat{x}$ so that if the initial $E(d|d > 0)$ is below $\hat{x}$ then average non-zero development expenditures are likely to increase (see the uniform and exponential cases in the first column of the upper panel in Table 2). By the same token, if the initial $E(d|d > 0)$ is above $\hat{x}$ then average non-zero development expenditures are likely to decrease (see the lognormal case). Whether $E(d|d > 0) \gtrless \hat{x}$ depends on the parameters of the model and on the distribution of x.

Changes in ϕ affect expected development expenditures not only through the induced change in n but also directly. An increase in ϕ, increases $C(x)$ at any x and

may also increase the maximizer $\widehat{x}$.[18] Thus, the firm aims at a higher $\widehat{x}$. In addition, $\overline{x}$ increases and this means that more expensive solutions will be undertaken. Thus, holding n constant, an increase in ϕ induces an increase in development expenditures, on average. As n increases, the effect on d due to the increase in $C(x)$ is strengthened when $E(d|d > 0) < \widehat{x}$, and weakened when $E(d|d > 0) > \widehat{x}$.

In short, given some form of strategic complementarity between n and ϕ (π), firms with higher current returns ϕ, or higher probability of success π, will have larger research departments (n) and will also develop their ideas more often. Consequently, they will make faster progress in their R&D program. The same is true for firms with a lower cost of setting up research teams b. These firms will be more technologically advanced than firms with a lower ϕ or π, or higher b and, even controlling for the technological level ω, these firms will be more valuable. The model, however, does not imply an unambiguous effect of these parameters on development expenditures d. Consequently, the effect of the model's parameters on total R&D expenditures–likely to be the only observed R&D measurement–is also ambiguous, and depends on the parameter configuration and distributional assumptions.

The dynamics implied by the model are quite simple. Starting from an initial state ω_0, ω remains constant at ω_0 until the cost of the selected project among the n draws $(x_1, \ldots, x_n)$ is below the reservation cost $\overline{x}$ at some time t_1. The R&D state next period is a random variable given by (1): it can advance to $\omega_0 + 1$ with probability $\pi(d)$ or remain at ω_0 with probability $1 - \pi(d)$. This process is repeated in every period. Over time, as the firm successfully completes R&D stages, ω increases in "jumps" of size one. Thus, if we take a cross-section of identical firms and let them evolve independently over time, some firms will randomly advance on their research programs while others will not. The different realizations of the R&D process creates heterogeneity across firms. This heterogeneity is purely due to chance but is persistent.

We close this section by examining in more detail the simulations underlying the results in Table 2. We focus on the evolution of ω and of R&D expenditures over time. The results are based on the following parameters: $\beta = 0.96$, $a_0 = 0.08$ and $a_1 = 0.10$, $b = 0.5$ and $\phi = 1$, using the uniform distribution, $G(x) = \frac{x}{80}$ (for illustrative purposes, the exponential and lognormal distribution are indistinguishable from the uniform distribution).

---

[18]But note that the numerical resutls in Table 2 are based on the piece-wise linear functional form for π that keeps $\widehat{x}$ constant at $\frac{1}{a_1} = 10$.

We know from Table 1 that the optimal number of research teams is 5. After initializing the ω process at $\omega_1 = 1$ we drew 5 $x's$ from the $U(0, 80)$ and selected the one x that gives the highest continuation value, $C(x^*)$. If $x^* \leq 19.2$ or, equivalently, if $C(x^*) \geq 0$, we set $d_1 = x^*$ and drew a Bernoulli variable representing "success" with probability $\pi(x^*)$ using (18). If a success resulted, we set $\omega_2 = \omega_1 + 1$; otherwise $\omega_2 = \omega_1$. Of course, when $x^* > 19.2$, the project was not developed and we set $d_1 = 0$ and $\omega_2 = \omega_1$. We simulated this process $T = 100$ times (periods) and generated a times series $\{\omega_t, d_t\}$. We replicated this simulation 3000 times. We can think of each replication as an independent R&D program or firm drawn from the same population.

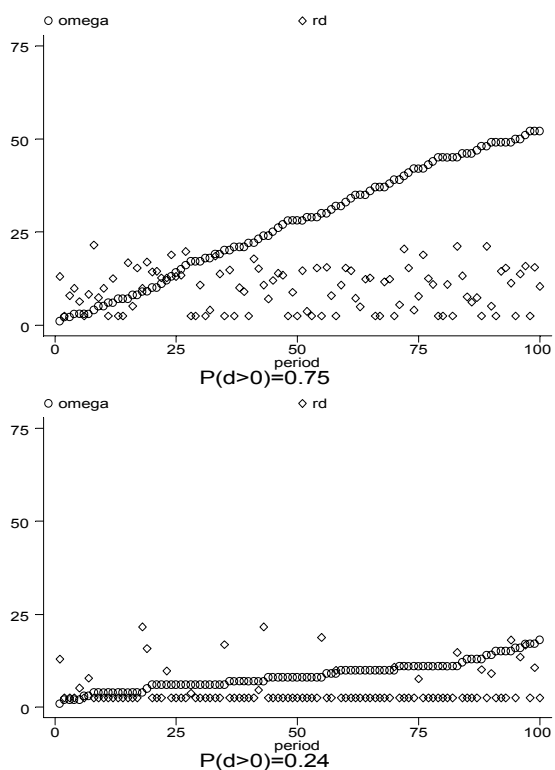Figure 4 plots the evolution of ω_t and R&D expenditures over time for a single firm.

Figure 4: ω and total R&D expenditures $(nb + d)$ over time

The time-path of ω is a step function with probability $\pi(d)F_n(\overline{x})$ of moving up. Because the mean of x is 40, it is very likely that at least one of the five draws will be below 19.2 $(1 - F_n(0) = 0.746)$. Thus, the project is developed quite often and its time-path looks quite smooth (top panel in Figure 4).

In order to generate a path that looks more like a step-function we need to lower the probability of development. We do this by increasing b to 1.1 which lowers n to 1 (bottom panel in Figure 4). Under the $U(0, 80)$ distribution, the probability that the single x draw is less than 19.2 is about a quarter. The bottom graph look more like a step function. In the top graph, the project is developed more frequently and therefore reaches a higher state at the end of the period.

Averaging across the 3000 replications and plotting this average against time, we would get a smooth straight line depicting the evolution of $E(\omega_t)$ over time.[19]

Per-period R&D expenditures $nb + d$ are also graphed. Recall that nb is fixed at $5 \times 0.5 = 2.5$ plus a random variable d given by (15). One of the features of the model is that firms do not develop their ideas continuously. The simulated firm in the top graph did R&D in 74 periods out of a 100 and succeeded in 51 occasions giving an average success rate of 69 percent. Table 3 reports averages across all 3000 firms of the fraction of periods involved in development, of the average R&D expenditures $nb + d$ and the final technology level at which they arrived (using the parameters of the top graph).

Table 3: Summary Statistics				
	Fraction of periods with development	Average R&D expenditure by firm	Final ω	Success rate
Mean (across 3000 firms)	0.746	9.71	47.41	0.622
STD	0.044	0.58	4.94	0.056

The success rate equals the last ω reached minus 1 divided by the number of times the firm did R&D.

As expected, firms are engaged in development in about 75 percent of their time and are successful at it on 62 percent of the trials. As a results they reach a final technological state of about 47 on average ($\approx 100 \times 0.75 \times 0.62$).

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[19]Roughly speaking, the value of ω after 100 periods in the top panel of Figure 5 should be equal to

$$\omega_1 + 99(1 - F_n(0))E(\pi(d)) =$$
$$1 + 99 \times 0.746 \times [0.08E(d|d < 10, d > 0)P(0 < d < 10) + .8P(d > 10)] =$$
$$1 + 99 \times 0.746 \times \left[0.08 \times 5.893 \times \frac{116460}{223659} + 0.8 \times \left(1 - \frac{116460}{223659}\right)\right] =$$
$$= \quad 47.45$$

using the simulated data to estimate the expectations and probabilities. This result is very close to the simulated value of ω after 100 periods averaged across the 3000 replications which is 47.41.

This average technological state masks a lot of variation across firms. Figure 5 graphs the evolution over time of five deciles of the cross-sectional distribution of ω.
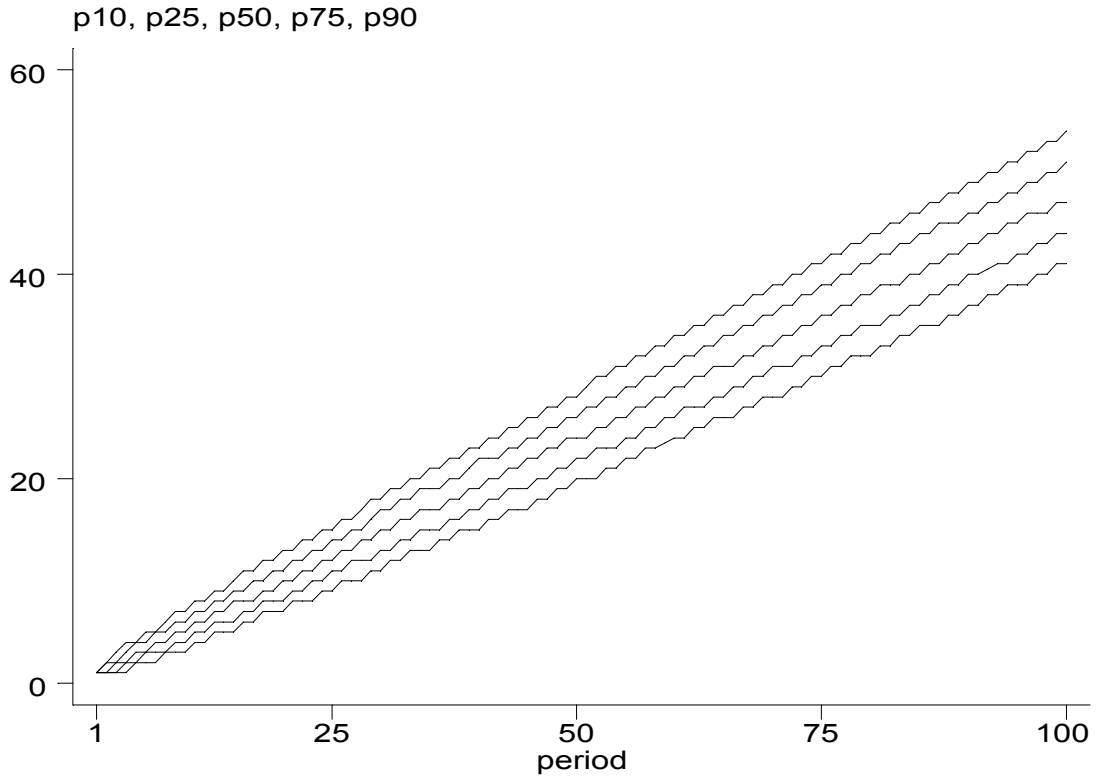
p10, p25, p50, p75, p90

Figure 5: 10th, 25th, 50th, 75th and 90th Percentiles of $\omega's$ Cross-sectional Distribution

In period 1, firms are all identical having the same initial $\omega_1 = 1$. As time passes and firms randomly advance on their research programs, the different realizations of the R&D process creates heterogeneity across firms. This heterogeneity is purely due to chance but is persistent. In period 100, the bottom 10 percent of the firms have an ω below 41 while the top 10 percent has an ω above 54.

3   R&D Subsidies

R&D subsidies are among the instruments used by governments to foster and guide technological change. In some countries, such as Israel, Spain, Finland, Norway and the U.S., governments directly subsidize the financing of firm-level R&D projects.[20] In this

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[20] For example, the R&D support given by the ATP in the U.S., by TEKES in Finland, by CDTI in Spain and by the Norwegian government operate in a somewhat similar manner.

section we analyze the effect of an R&D subsidy by focusing on the effects of a subsidy to the development phase in the R&D program.[21]

We assume that application costs are nil. This guarantees that the firm applies for a subsidy in every development phase it reaches. The sequence of events is now as follows. The firm first decides on the number of research teams n. It then samples n $x's$ and submits a proposal for a subsidy specifying an expected development cost.[22] The proposal is evaluated by the agency granting the subsidies and a response reaches the firm immediately. The firm then computes the continuation values of the n $x's$ and decides on $x^*-$the most profitable solution–given knowledge on the subsidy status of the project.

Receiving a development subsidy is an uncertain event and we assume that the probability of receiving it, denoted by λ, does not depend on the proposed estimated cost of the project nor on the firm's technological state ω. The subsidy is a fraction $0 \leq \alpha \leq 1$ of the actual project cost.[23] If the subsidy is received, the development cost to the firm is $(1-\alpha)x$. If the subsidy is not received the cost remains x. The cost of the R&D project can thus be written as $(1-\alpha s)x$, where $s = 1$ when the subsidy is received and $s = 0$, otherwise.

The presence of development subsidies changes the continuation value associated with a project costing x, equation (4). We now need to condition on the presence of the subsidy. Omitting the $\omega's$, the continuation value is

$$C(x,s) = \beta \pi(x) \left[V(\omega + 1) - V(\omega) \right] - (1-\alpha s)x \tag{19}$$

When $s = 1$, the continuation curve is shifted upwards so that the reservation cost increases. The reservation cost $\overline{x}(s)$ is therefore an increasing function of s. A simple comparison of derivatives shows that $\widehat{x}(s)$, the maximizer of $C(x,s)$, shifts to the right, while the smallest root of $C(x,s)$ decreases and its largest root increases (see Figure 6).

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[21]Subsidies to research can be thought as reductions in b.

[22]The expected cost can be based on the n $x's$ in the sample, on the distribution function of x^*, or on some other estimate of the development costs. This will not matter in what follows.

[23]In Israel, the major requirement to qualify for a subsidy is that the project be "technologically feasible". Usually, α is 50 percent. For an excellent account of R&D policy in Israel see Trajtenberg (2000).
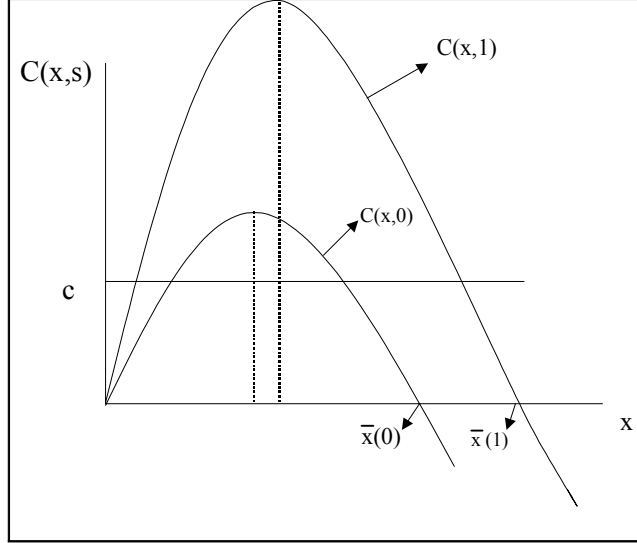
Figure 6 : Continuation Value and Subsidies

Because the firm selects x^* only after its subsidy status is realized, the level of development expenditures also changes with s. Let (x_0^*) x_1^* denote the x^* when (no) subsidy is received. It is intuitively clear that $x_1^* \geq x_0^*$.[24]

The only difference with the model in Section 2 is that since $C(\cdot)$ is also a function of s, the expectation in the functional equation (6) is now taken over s as well. Using iterated expectations, and proceeding as before, we can easily show that the value function now satisfies

$$(1 - \beta) V(\omega) = \underset{n}{Max} \phi\omega - nb + (1 - \lambda) \left(\int_0^{\widehat{C}(0)} cF_n(c,0)dc \right) + \lambda \left(\int_0^{\widehat{C}(1)} cF_n(c,1)dc \right)$$

(20)

where $\widehat{C}(s) = \frac{\beta\pi(\widehat{x}(s))\phi}{(1-\beta)} - (1 - \alpha s)\widehat{x}(s)$, $\widehat{x}(s)$ satisfies $\frac{\phi\beta\pi'(\widehat{x}(s))}{1-\beta} = 1 - \alpha s$, and

$$F_n(c,s) = \left[1 + G(x^-(c,s)) - G(x^+(c,s))\right]^n$$

where $x^-(c,s)$ and $x^+(c,s)$ are, respectively, the minimal and maximal roots of the equation $C(\omega, x, s) = c$. Note that the model in Section 2 corresponds to the special case where $\lambda = 0$.

---

[24]Note that $C(\omega, x, s = 1) = C(\omega, x, s = 0) + \alpha x$. Thus, in a sample of n $x's$, the x_i that maximizes $C(\omega, x, s = 1)$ cannot be smaller than the one that maximizes $C(\omega, x, s = 0)$ because at a lower x_i the values of both $C(\omega, x, s = 0)$ and αx would be decreased.

27

Importantly, the linearity of $V(\omega)$ is preserved. The marginal benefit of a research team is now the difference in the conditional expected continuation value–the last two terms in (20) as we move from $n-1$ to n research teams

$$
\begin{aligned}
MB_n(\lambda) &= (1-\lambda)\,MB_n(0) + \lambda MB_n(1) \\
&= MB_n(0) + \lambda\left(MB_n(1) - MB_n(0)\right)
\end{aligned}
\tag{21}
$$

where $(MB_n(0))\,MB_n(1)$ is the marginal benefit given that a subsidy was (not) received.

The analysis of the effects of subsidies is divided into two parts. First, we analyze the effect of receiving a subsidy on the firm's R&D activities, i.e., the effect of switching s from $s=0$ to $s=1$, and then show the effects of changing the subsidy parameters λ and α.

3.1 Effects of Receiving a Subsidy

Note that the assumption on the timing of events implies that the amount of research n is not affected by the actual realization of s but it is affected the *expected* value of s, λ. Put differently, the marginal benefit of n does not depend on s. Thus, receiving a subsidy can only affect the amount of development.

It is immediately clear from Figure 6 that the reservation cost project increases when a subsidy is received. Thus, receiving a subsidy increases the likelihood of developing the project but does not affect research efforts. Thus, one type of *subsidy effect* is the increase in the probability of development,

$$
\begin{aligned}
\Delta_P &\equiv P(d > 0 | s = 1) - P(d > 0 | s = 0) \\[2ex]
&= P(x_1^* < \overline{x}(1) | s = 1) - P(x_0^* < \overline{x}(0) | s = 0) \\[2ex]
&= [1 - G(\overline{x}(0))]^n - [1 - G(\overline{x}(1))]^n
\end{aligned}
$$

using (10).

Clearly, Δ_P is positive because $\overline{x}(1) > \overline{x}(0)$.[25]

---

[25] Also, because increases in α increase $\overline{x}(1)$, but not $\overline{x}(0)$, the probability of development increases when a subsidy is received, and remains unchanged when a subsidy is *not* received. Thus, Δ_P increases with α for given n. Changes in λ also affect Δ_P but only through the change that is induced on n. As

Of greater interest, perhaps, is the effect of the subsidy on the level of R&D expenditures. The effect of receiving a subsidy on R&D expenditures depends on the project's realized cost x^*. Recall that $x_1^* \geq x_0^*$. Clearly, receipt of a subsidy leads to higher R&D expenditures if x_0^* is above the reservation values $\overline{x}(0)$, but x_1^* is below $\overline{x}(1)$. The R&D stage would not have been undertaken without the subsidy but will be undertaken if a subsidy is received. Hence, R&D expenditures change from nb to $nb + x_1^*$.

On the other hand, if $x_0^* > \overline{x}(0)$ and $x_1^* > \overline{x}(1)$ receipt of a subsidy does not change the firm's decision not to implement the project; R&D expenditures remain at nb.[26]

Finally, when the realized cost satisfies $x_0^* \leq \overline{x}(0)$, development of the project once a subsidy is received must also be profitable since its costs are decreased. This implies $x_1^* \leq \overline{x}(1)$. The firm's change in total R&D expenditures is $x_1^* - x_0^* \geq 0$ because development would have been undertaken at a cost x_0^* even if the subsidy was not received. The firm, of course, profits from the subsidy because its R&D costs are reduced by αx_1^* and this increases its continuation value. Because the agency deciding on the subsidies usually does not know the firm's $\overline{x}'s$, the subsidy can be in effect quite unnecessary for those projects with costs below $\overline{x}(0)$.

In short, the above counterfactual comparison of R&D expenditures for a subsidized project implies that the change in total R&D expenditures is either $x_1^* - x_0^*$ when $x_0^* \leq \overline{x}(0)$ or x_1^* when $x_0^* > \overline{x}(0)$ and $x_1^* < \overline{x}(1)$. Otherwise, the change is zero.

In applied work, however, one usually compares the average R&D expenditures of subsidized to non-subsidized projects,

$$\Delta_{R\&D} \equiv E(nb + d|s = 1) - E(nb + d|s = 0)$$

$$= \int_0^{\overline{x}(1)} x g_{x_1^*}(x; s = 1, n) dx - \int_0^{\overline{x}(0)} x g_{x_0^*}(x; s = 0, n) dx \tag{22}$$

$$= \int_0^{\overline{x}(0)} x \left[g_{x_1^*}(x; s = 1, n) - g_{x_0^*}(x; s = 0, n) \right] dx + \int_{\overline{x}(0)}^{\overline{x}(1)} x g_{x_1^*}(x; s = 1, n) dx$$

---

n increases, $\overline{x}(1)$ remains fixed but the probability that $x_1^* \leq \overline{x}(1)$ increases. However, the probability that $x_0^* \leq \overline{x}(0)$ also increases with increases in n. The effect of changes in λ on Δ_P is thus ambiguous and depends on the parameter configuration.

[26]If a subsidy is received but the cost of the project is so high that it is not profitable to undertake development, the firm can rescind the right to use the subsidy at no additional cost.

where $g_{x^*}(\cdot)$ is the density of x_0^* or x_1^* given in (16).[27]

The first term captures the increase in average development expenditures of those projects that would have been undertaken without the subsidy (the average of $x_1^* - x_0^*$) while the second term represents the average development expenditures of those projects that would not have been undertaken without the subsidy.[28]

Of course, $\Delta_{R\&D} \geq 0$ because new projects are undertaken when the subsidy is received and even the projects that would have been developed without the subsidy are undertaken at a higher level of expenditure. Moreover, because $\overline{x}(1)$ increases with α, $\Delta_{R\&D}$ increases with the subsidy share.

A natural question that arises in the analysis of R&D subsidies is their "additionality" effect. Does the subsidy increase total R&D expenditures by more than the amount of the subsidy? In other words, do own (company-financed) R&D expenditures increase after receiving the subsidy?

The answer is not obvious because even though there are more projects being implemented, firms spend only a fraction of what they would have spent on those projects with costs below the "no-subsidy" reservation cost $\overline{x}(0)$. The change in *own* R&D caused by the subsidy to a project with costs x_0^* is $(1 - \alpha)x_1^*$ when $x_0^* > \overline{x}(0)$ and $x_1^* < \overline{x}(1)$. On the other hand, when $x_0^* \leq \overline{x}(0)$, the change in own R&D expenditures equals $(1 - \alpha)x_1^* - x_0^*$ which can be either positive or negative.[29] Comparing, the average own R&D expenditures of the subsidized projects to that of the non-subsidized projects we obtain,

---

[27]The density of x_1^* uses the continuation function (19) evaluated at $s = 1$. The averages are computed using the population of projects. If we wish to restrict the universe only to projects engaged in development we need to divide the density of x^* by the probability of development.

[28]Because subsidies are randomly assigned in the model, the simple difference in average R&D by susbidy status is the same as the average effect of the subsidy on the subsidized firms.

[29]When $x_0^* > \overline{x}(0)$ and $x_1^* > \overline{x}(1)$, the change in own R&D is zero.

$$\Delta_{R\&D}^{own} \equiv E(nb + (1-\alpha)d|s=1) - E(nb + d|s=0)$$

$$(23)$$

$$= \int_0^{\overline{x}(0)} x \left[g_{x_1^*}(x; s=1, n) - g_{x_0^*}(x; s=0, n) \right] dx$$

$$- \alpha \int_0^{\overline{x}(0)} x g_{x_1^*}(x; s=1, n) dx + (1-\alpha) \int_{\overline{x}(0)}^{\overline{x}(1)} x g_{x_1^*}(x; s=1, n) dx$$

$$= \Delta_{R\&D} - \alpha E(d|s=1)$$

The last term in the third row of (23) is the increment to own R&D expenditures in the projects undertaken as a direct result of receiving the subsidy while the second term captures the decrease in own R&D due to the subsidization of inframarginal projects, projects that would have been undertaken even without the subsidy. The subsidy effect on own R&D expenditures will be positive when α is not "too large".

To gain more understanding on the forces impinging on the additionality effect we simulated the effects of receiving a subsidy using the same parameters of the model as in the previous set of simulations in Section 2.4, except that $b = 1.1$ as in the bottom graph of Figure 4.[30] The probability of receiving a subsidy is set at $\lambda = 0.60$ in the top panel, which is close to the reported 70 percent of all applications receiving a subsidy in Israel (Trajtenberg, 2000). In the bottom panel, the subsidized fraction of the costs is set to a half reflecting also the typical subsidy in Israel.

The last two columns in Table 4 show simulated values of $\Delta_{R\&D}$ from (22) and of the additionality effect $\Delta_{R\&D}^{own}$ from (23). Note that the effect of the subsidy on own R&D is an order of magnitude smaller than its effect on total R&D expenditures. In Panel A we observe that, given λ, $\Delta_{R\&D}^{own}$ increases and then decreases with α. This reflects the trade-off between the positive effect of a higher α on the average change in total R&D expenditures ($\Delta_{R\&D}$) and the negative effect on *own* R&D expenditures ($-\alpha E(d|s=1)$). Panel B shows a similar pattern for changes in λ.

---

[30] $\beta = 0.96$, $a_0 = 0.08$ and $a_1 = 0.10$, and $\phi = 1$. The uniform distribution is assumed, $G(x) = \frac{x}{80}$.

		Table 4: Subsidy Effects			

		A: Changes in α, $\lambda = 0.60$			
α	n	$P(d>0\|s=1)$	$E(n_\lambda b + d\|s=1)$	$\Delta_{R\&D}$	$\Delta_{R\&D}^{own}$
0.10	1	0.27	3.94	0.52	0.24
0.20	2	0.51	8.25	1.99	0.78
0.30	2	0.57	9.82	3.56	1.28
0.40	3	0.78	15.08	6.38	1.67
0.50	3	0.86	18.19	9.50	2.05
0.60	3	0.94	22.25	13.55	2.18
0.70	3	0.99	26.60	17.91	1.59

		B: Changes in λ, $\alpha = 0.50$			
λ	n	$P(d>0\|s=1)$	$E(n_\lambda b + d\|s=1)$	$\Delta_{R\&D}$	$\Delta_{R\&D}^{own}$
0.10	1	0.48	10.30	6.90	2.30
0.20	2	0.73	15.48	9.22	2.58
0.30	2	0.73	15.50	9.24	2.59
0.40	3	0.86	18.19	9.48	2.04
0.80	3	0.86	18.19	9.49	2.05
0.90	4	0.93	19.70	8.87	1.22

3.2   Effects of Changing Subsidy Parameters

Table 4 also shows the effect of varying the subsidy parameters λ and α on the amount of research and on the probability of development (columns (2) and (3)). The effect of λ on n depends on the sign of $MB_n(1) - MB_n(0)$ (see (21)). As was the case when analyzing the effect of ϕ on n, the marginal benefit of research does not necessarily increase when a subsidy is received, even though the continuation value does. This depends on $G(x)$ and on the success probability function $\pi(x)$. As before, it can be shown that a sufficient condition that ensures monotonic comparative statics is that $1 - F_n(c; s)$ exhibits strictly increasing differences in (n, s).

The simulation results in Panel B of Table 4 show that n is nondecreasing in λ. That is, higher probabilities of being subsidized increase the optimal number of research

teams.[31] Increases in λ lead to increases in n which lead to increases in the proportion of projects engaged in development simply because more projects become profitable, i.e. more projects set $\overline{x}(1)$ as their reservation cost. This also increases R&D expenditures of the subsidized projects (Column 4). A similar pattern is observed in Panel A when α varies but note that R&D expenditures increase at an increasing rate with changes in α. This occurs as changes in α push out $\overline{x}(1)$. Thus, for a given probability of subsidization, changes in α induce increasingly more expensive projects to become profitable to the firm.[32]

4 R&D and Productivity Growth

An interesting aspect of the model is the evolution of ω and of R&D expenditures over time and the relationship between them (see Figure 5). It is intuitively clear that the more frequently the R&D stages are implemented, the larger ω will, on average, be. The connection between the level of R&D expenditures and the technology state, however, is less obvious.

Conditional on ω_t, the probability that ω increases (by one) is

$$
\begin{aligned}
P(\omega_{t+1} - \omega_t = 1 | \omega_t) = P(d > 0)\pi(d_t) &= \left(1 - [1 - G(\overline{x})]^n\right)\pi(d_t) \\
&\equiv p(n, d_t)
\end{aligned}
$$

because the events $\{x^* \leq \overline{x}\}$ and $\{technical\ success\}$ are independent.

The probability of overall success $p(n, d)$ embeds both the probability of implementation and the probability of the purely technical success of the implemented solution. The expected value of ω_{t+1} given ω_t, n and d_t is

$$
\begin{aligned}
E(\omega_{t+1} | \omega_t, n, d_t) &= \omega_t(1 - p(n, d_t)) + (\omega_t + 1)\, p(n, d_t) \qquad (24)\\
&= \omega_t + p(n, d_t)
\end{aligned}
$$

---

[31]The discontinuous increase in n is a consequence of the fact that the optimal number of research teams must be integer-valued. Notice that there are also discontinuous jumps in the proportion of firms engaging in development when the optimal number of research teams changes.

[32]As in precious cases, n does not necessarily increase with α. It will do so is if $1 - F_n(c; \alpha)$ exhibits strictly increasing differences in (n, α).

Note that when $d = 0$ we have $p(n, 0) = 0$ because $\pi(0) = 0$, so that the R&D program does not advance when the solutions are not developed. From (24), next period technological state depends on today's state and on the probability of developing the proposed solution and of succeeding. The latter probabilities increase, respectively, with research and development expenditures so that, ceteris paribus, firms spending more on R&D should be more advanced in their R&D program.

The connection between the technological state ω and output or total factor productivity (TFP) is made through the assumption that ω affects the productivity of capital in a Cobb-Douglas production function, and that capital is, for simplicity, fixed at $\overline{K}$,

$$y = \left(\omega \overline{K}\right)^\gamma L^{1-\gamma} \tag{25}$$

Given ω, there is no interaction between the choice of L and the R&D policy so these two aspects of the optimal firm policy can be analyzed separately. The profit of a price-taking firm, gross of its R&D expenditures, is linear in ω,

$$profit = \underbrace{p^{\frac{1}{\gamma}} p_l^{1-\frac{1}{\gamma}} (1-\gamma)^{\frac{1}{\gamma}} \frac{\gamma}{1-\gamma} \overline{K}}_{\phi} \omega = \phi \omega$$

where p is the price of the final product and p_l is the wage rate.

Observe from (25) that the level of total factor productivity is ω^γ so that, using (24), we obtain

$$
\begin{aligned}
E(\ln TFP_{t+1} | \omega_t, n, d_t) &= E(\gamma \ln \omega_{t+1} | \omega_t, n, d_t) \\
&= \gamma \ln \omega_t (1 - p(n, d_t)) + \gamma \ln (\omega_t + 1) \, p(n, d_t) \\
&= \gamma \ln \omega_t + \gamma \ln \left(1 + \frac{1}{\omega_t}\right) p(n, d_t) \\
&\approx \ln TFP_t + \frac{\gamma p(n, d_t)}{\omega_t}
\end{aligned}
$$

The expected growth rate in TFP, conditional on lagged TFP level and on research and development expenditures, is

$$
\begin{aligned}
E\left(\Delta \ln TFP_{t+1} | \omega_t, n, d_t\right) &= \frac{\gamma}{\omega_t} p(n, d_t) \tag{26} \\
&= \phi \frac{p(n, d_t)}{s_t}
\end{aligned}
$$

using sales $s_t = p y_t = \frac{\phi}{\gamma} \omega_t$.

Note that because success probability function $p(\cdot)$ is bounded by one doubling R&D expenditures will not always double expected output. Thus, this model echoes Jones's (1995) critique of the presence of "scale effects"–that the growth rate of the economy is proportional to the size of the resources devoted to R&D–in R&D-based growth models.

In particular, because of the fixed unit increment in ω and the boundedness of $p(\cdot)$, the growth rate in the technology state of the more advanced firms should be lower, on average, than that of the less advanced firms, controlling for R&D expenditures. In other words, firms grow faster when they are smaller (in terms of ω or sales) implying some convergence in technology levels as in the model of Jovanovic and MacDonald (1994) but for different reasons.

We can also use the model to interpret the established empirical tradition of regressing total factor productivity (TFP) growth on an R&D-to-sales ratio and other regressors,

$$\Delta \ln TFP = \rho \left(\frac{R\&D}{Sales} \right) + controls + error \tag{27}$$

The parameter ρ is interpreted as the rate of return to R&D. The conceptual framework underlying this procedure, presented in Griliches (1979), relies on the existence of a production function relating output to classical inputs and to "knowledge". Knowledge, in turn, is the result of past investments in R&D thereby establishing a link between R&D expenditures and output (sales). The estimated rates of return to R&D are, in general, significantly positive (see Griliches, 1998) and large (the selected estimates in Jones and Williams (1998) are between 30 and 100 percent).

Our model provides a sounder theoretical basis for this empirical approach, emphasizing the within-firm dimension of technological progress and complements recent work on the social rate of return to R&D obtained in a growth model where knowledge spillovers, congestion externalities and creative destruction are allowed (Jones and Williams, 1998).

The question we want to address is the following: In light of the population regression equation (26), what is the parameter ρ in the estimated regression equation (27) capturing? Suppose we have a sample of R&D-doing firms–firms that are doing research *and* development. Then,

$$E\left(\Delta \ln TFP_{t+1} | \omega_t, n, d_t > 0 \right) = \phi \frac{\pi(d_t)}{s_t} \tag{28}$$

because $x^* < \overline{x}$ with probability 1 for these firms.

Using a linear approximation to $\pi(d)$ around $E(d)$ in (28) we obtain,

$$
\begin{aligned}
E\left(\Delta \ln TFP_{t+1} | \omega_t, n, d_t > 0\right) &= \phi \frac{\pi(E(d))}{s_t} + \phi \frac{\pi'(\widetilde{d})}{s_t}(d_t - E(d)) \\
&= \phi \frac{\pi(E(d)) - \pi'(\widetilde{d})\left[nb + E(d)\right]}{s_t} + \phi \frac{\pi'(\widetilde{d})}{s_t}\left[nb + d_t\right] \\
&= \varphi \frac{1}{s_t} + \underbrace{\phi \pi'(\widetilde{d})}_{\rho}\left(\frac{R\&D}{sales}\right)_t
\end{aligned}
$$

where $\widetilde{d}$ is a point between d_{t-1} and $E(d)$ and $\varphi = \phi\pi(E(d)) - \pi'(\widetilde{d})\left[nb + E(d)\right]$.

Recall that the continuation value is $C(x) = \frac{\beta\pi(x)\phi}{(1-\beta)} - x$. Thus $\rho = \phi\pi'(\widetilde{d})$ reflects the short-run (one-period, not discounted) gross return to development expenditures.[33]

5   Conclusions and Extensions

This paper attempts to penetrate the "black box" of technology making. We develop a dynamic model of R&D at the project level and draw testable implications from it. A main feature of the model is the clear distinction between research and development activities. Research generates ideas with different costs and probabilities of success, development implements the most profitable among them. This formulation of the R&D process produces a non-linear relationship between R&D inputs and output: when the cost of development is too high, the firm postpones development. Research, however, is always conducted with the hope of generating profitable, implementable ideas.

The model also provides a theoretical framework for analyzing the impact of R&D subsidies and illustrates that the only meaningful magnitude to analyze is the "additionality effect": the change in the amount of company-financed R&D caused by the subsidy. Whether this effect is positive or negative depends on the parameters and distributional assumptions. It is, therefore, an empirical matter. An additional important feature of the model is that can be used to cast light on the R&D-productivity relationship and to interpret anew the meaning of the parameter usually estimated in micro-level productivity regressions.

---

[33]Note also that one of control variables in the estimated regression should be (inverese) sales or size.

Although the present model imposes several simplifying assumptions, it captures many salient features of R&D activities. The limitations of the model can, of course, be overcome by extending the model in several directions. First, the model implies that sales are one-to-one with the technological state ω and, therefore, monotonically increase over time. There are no fluctuations in sales. This can be amended by abandoning the assumption of a known demand (prices) and allowing current period returns, or ϕ, to be stochastic.

Second, the amount of research n does not depend on the state ω. This can be overcome by specifying the probability of successful development π to depend also on the technological state, i.e., $\pi(x, \omega)$. Assuming positive but decreasing marginal returns to ω will capture a learning-by-doing effect. That is, success comes easier to more successful firms but after the easier stages have already been implemented, the R&D technical problems become harder and the probability of success increases at a decreasing rate (see Kortum, 1997; Bental and Peled, 1996). In this case, the reservation cost is likely to increase with ω (at least up to some level of ω) implying more development expenditures as the project grows over time. On the other hand, firms will invest more in research n at the early stages of their projects because the marginal increase in π is larger the smaller is ω. The learning-by-doing assumption, therefore, will produce richer dynamics.

Another interesting extension of the model is to endogenize the cost of research b by introducing a labor market for scientists and engineers. The parameter b will then reflect the equilibrium wages in this market. The implications of this general equilibrium extension can be significant when analyzing the effect of subsidies that stimulate R&D because additional R&D puts pressure on the market for scientists and engineers leading to increases in their wages. These increases in wages can lead to a reduction in R&D activity thereby undoing part of the initial R&D subsidy effect.

Further, the probability of receiving a subsidy can be made to depend on the development expenditures x^* and on the technological state ω. While doing this will impart a dose of realism to the model, the nature of the relationship between the subsidy probabilities and (previous) successes, size, or proposed budget is far from obvious and there is not much empirical guidance on this matter. In any case, this extension will allow researchers to deal with issues such as the effectiveness of R&D subsidies to "small" and "large" firms, to successful or unsuccessful ones, etc.

Lastly, the issue of spillovers between research teams *within* the firm (program) has been deliberately ignored. This is undoubtedly important as it bears upon the internal

organization of research at the firm. Spillovers *across* firms is also crucial not only for the empirical evaluation of the benefits of the subsidy program, but also for the design of the optimal subsidy parameters: in the absence of positive externalities across firms it is best not to have a subsidy program at all. Thus, the next step should be to introduce spillovers into the model.

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Appendix 1: Distribution of $C(\omega, x^*)$

Because $C(\omega, x^*)$ is the nth order statistic from the random sample $\{C_1, \dots, C_n\}$ where $C_i = C(\omega, x_i)$, its distribution function is

$$F_{\omega,n}(c) \equiv P(C^* \le c) = P(C_i \le c)^n$$

where

$$C^* \equiv C(\omega, x^*)$$

From Figure 1 in the text we observe that the distribution of C_i can be expressed as

$$
\begin{aligned}
P(C_i \le c) &= \int_{\{x:C(\omega,x)\le c\}} g(x)dx \\
&= \int_0^{x^-(c)} g(x)dx + \int_{x^+(c)}^{\infty} g(x)dx \\
&= G(x^-(c)) + 1 - G(x^+(c))
\end{aligned}
$$

where $g(x)$ is the density of x and $x^-(c)$ and $x^+(c)$ are, respectively, the minimal and maximal roots of the equation $C(\omega, x) = c$. Note that $x^-(c) \le \widehat{x}(\omega) \le x^+(c)$.

We then have,

$$F_n(c) = \left[1 + G(x^-(c)) - G(x^+(c))\right]^n$$

Appendix 2: Proof of Proposition 1

Let $W(\omega) = A\omega + B$ be a candidate value function. We will check that, for some parameters A and B, $W(\omega)$ satisfies the functional equation (6). From (4), the continuation value is

$$C(\omega, x) = \beta\pi(x)\left[W(\omega + 1) - W(\omega)\right] - x$$
$$= \beta\pi(x)A - x$$

which we write as $C(\omega, x) = C(x)$ because it does not depend on ω,

The maximizer of $C(x)$ satisfies $\beta\pi'(\widehat{x})A = 1$, and we have $\widehat{C} = \beta\pi(\widehat{x})A - \widehat{x}$, while $x^-(c)$ and $x^+(c)$ are, respectively, the minimal and maximal roots of $\beta A\pi(z) - z - c = 0$, which do not depend on ω. Using these values of $\widehat{C}$, $x^-(c)$ and $x^+(c)$ note that the solution value of n to the following problem

$$\underset{n}{Max} \; \frac{1}{1-\beta}\left(-nb + \int_0^{\widehat{C}} c\left[1 + G(x^-(c)) - G(x^+(c))\right]^n dc\right) \equiv B \qquad (29)$$

does not depend on ω (but depends on A) because the value of the integral does not depend on ω.

Defining B as the maximum value in (29) and $A = \frac{\phi}{1-\beta}$ we have that the function $W(\omega)$ satisfies

$$
\begin{aligned}
W(\omega) &= A\omega + B \\
&= \frac{\phi\omega}{1-\beta} + \underset{n}{Max} \; \frac{1}{1-\beta}\left(-nb + \int_0^{\widehat{C}} c\left[1 + G(x^-(c)) - G(x^+(c))\right]^n dc\right) \\
&= V(\omega)
\end{aligned}
$$

with $C(x) = \frac{\phi\pi(x)\omega}{1-\beta} - x$.

Appendix 3: Proof of Proposition 2

P roof. Rewrite the value function in Proposition 1 as

$$V(\omega) = \underset{n}{Max} W(\omega, n),$$

$$W(\omega, n) = \frac{1}{1-\beta} \left(\phi\omega - nb + \int_0^{\widehat{C}} cF_n(c)dc \right)$$

$$= \frac{1}{1-\beta} \left(\phi\omega - nb + \widehat{C} - \int_0^{\widehat{C}} F_n(c)dc \right)$$

Using an "envelope theorem" type of argument we obtain

$$(1-\beta)\frac{\partial V(\omega)}{\partial \phi} = (1-\beta)\frac{\partial W(\omega, n^*)}{\partial \phi} = \omega + \frac{\partial \widehat{C}}{\partial \phi} - \frac{\partial \widehat{C}}{\partial \phi}F_{n^*}(\widehat{C}) - \int_0^{\widehat{C}} \frac{\partial F_{n^*}(c)}{\partial \phi}dc$$

$$= \omega - \int_0^{\widehat{C}} \frac{\partial F_{n^*}(c)}{\partial \phi}dc$$

$$= \omega + \int_0^{\widehat{C}} n^* F_{n^*-1}(c)\frac{\partial H(c)}{\partial \phi}dc$$

using (9), where n^* is the optimal n and $0 \leq H(c) = G(x^+(c)) - G(x^-(c)) \leq G(\overline{x})$ is decreasing in c.

As ϕ increases, the curve $C(x)$ shifts up. At any given c, therefore, the smallest root of the equation $C(x) - c = 0$ decreases whereas the largest root increases, i.e., $x^-(c)$ decreases while $x^+(c)$ increases. Correspondingly, $H(c)$ increases with ϕ and therefore $\frac{\partial V(\omega)}{\partial \phi} > 0$. Analogously, we have $\frac{\partial V(\omega)}{\partial \pi} > 0$. ∎

Appendix 4: Parameter Monotonicity

The expected continuation value was defined in (7) as

$$EC(n, \phi) = E\left(C(x^*)|C(x^*) \geq 0, n, \phi\right)(1 - F_n(0))$$

omitting the argument ω and using (10).

Definition 1 *The function $EC(n, \phi)$ exhibits strictly increasing differences in (n, ϕ) if for all pairs (n_1, ϕ_1) and (n_2, ϕ_2), and given ω, it is the case that $n_2 \geq n_1$ and $\phi_2 \geq \phi_1$ imply*

$$EC(n_2, \phi_2) - EC(n_1, \phi_2) > EC(n_2, \phi_1) - EC(n_1, \phi_1) \tag{30}$$

Lemma 1 *If $EC(n, \phi)$ exhibits strictly increasing differences in (n, ϕ), then n is a non-decreasing function of ϕ.*

P roof. Let n_1 be optimal for ϕ_1, and let n_2 be optimal for ϕ_2. Optimality means that

$$n_i = \arg \underset{n}{Max} \; (\phi_i \omega - nb + EC(n, \phi_i))$$

for $i = 1, 2$. Let the objective function–the term in parenthesis–be $W(\omega, n, \phi)$. Optimality implies

$$W(\omega, n_2, \phi_2) - W(\omega, n_1, \phi_2) \geq 0 \geq W(\omega, n_2, \phi_1) - W(\omega, n_1, \phi_1)$$

which implies

$$EC(n_2, \phi_2) - EC(n_1, \phi_2) \geq EC(n_2, \phi_1) - EC(n_1, \phi_1) \tag{31}$$

Without loss of generality let $\phi_2 > \phi_1$. Suppose $n_2 < n_1$. Strictly increasing differences imply

$$EC(n_1, \phi_2) - EC(n_2, \phi_2) > EC(n_1, \phi_1) - EC(n_2, \phi_1)$$

which directly contradicts (31). Thus, $n_2 \geq n_1$ ∎

Lemma 2 *A sufficient condition for $EC(n, \phi)$ to exhibit strictly increasing differences in (n, ϕ) is that $1 - F_n(c; \phi)$ exhibits strictly increasing differences in (n, ϕ).*

P roof. Let $n_2 \geq n_1$ and $\phi_2 \geq \phi_1$.

$$EC(n_2, \phi_2) - EC(n_1, \phi_2) > EC(n_2, \phi_1) - EC(n_1, \phi_1)$$

$$\Longleftrightarrow EC(n_2, \phi_2) - EC(n_2, \phi_1) > EC(n_1, \phi_2) - EC(n_1, \phi_1)$$

$$\Longleftrightarrow \int_0^{\widehat{C}(\phi_1)} F_{n_2}(c, \phi_1)dc - \int_0^{\widehat{C}(\phi_2)} F_{n_2}(c, \phi_2)dc > \int_0^{\widehat{C}(\phi_1)} F_{n_1}(c, \phi_1)dc - \int_0^{\widehat{C}(\phi_2)} F_{n_1}(c, \phi_2)dc$$

$$\Longleftrightarrow \int_{\widehat{C}(\phi_1)}^{\widehat{C}(\phi_2)} \left\{ F_{n_1}(c, \phi_2) - F_{n_2}(c, \phi_2) \right\} dc$$

$$+ \int_0^{\widehat{C}(\phi_1)} \left[\left\{ F_{n_2}(c, \phi_1) - F_{n_2}(c, \phi_2) \right\} - \left\{ F_{n_1}(c, \phi_1) - F_{n_1}(c, \phi_2) \right\} \right] dc > 0$$

Note that the first integral is always positive because $F_n(c)$ decreases with n. A sufficient condition for the second integral to be positive is for the integrand to be positive at every c. This occurs whenever

$$\left\{ F_{n_2}(c, \phi_1) - F_{n_2}(c, \phi_2) \right\} - \left\{ F_{n_1}(c, \phi_1) - F_{n_1}(c, \phi_2) \right\} > 0$$

$$\Longleftrightarrow \left\{ F_{n_1}(c, \phi_2) - F_{n_2}(c, \phi_2) \right\} - \left\{ F_{n_1}(c, \phi_1) - F_{n_2}(c, \phi_1) \right\} > 0$$

$$\Longleftrightarrow \left\{ [1 - F_{n_2}(c, \phi_2)] - [1 - F_{n_1}(c, \phi_2)] \right\} > \left\{ [1 - F_{n_2}(c, \phi_1)] - [1 - F_{n_1}(c, \phi_1)] \right\}$$

which is equivalent to $1 - F_n(c; \phi)$ exhibiting increasing differences in (n, ϕ) ∎

Note that this is just a sufficient condition.